T2MATHS IS A PREMIUM COACHING SCHOOL DELIVERING THE HIGHEST QUALITY TEACHING FOR SECONDARY SCHOOL MATHEMATICS. OUR INSTRUCTORS DRAW FROM EXTENSIVE TEACHING EXPERIENCE IN SUBJECTS INCLUDING: VCE MATHEMATICAL METHODS, VCE SPECIALIST MATHEMATICS, IB MATHEMATICS AND UNIVERSITY ENHANCEMENT MATHEMATICS. T2MATHS INSTRUCTORS ARE MATHEMATICALLY ACCOMPLISHED, ENGAGING, AND CURRENTLY TEACH AT TOP INDEPENDENT SCHOOLS, SUPPORTING STUDENTS TO ACHIEVE OUTSTANDING RESULTS.

SAMPLE VCE METHODS EXAM 1 & EXAM 2 SOLUTIONS

EXAM 1 SOLUTIONS

Question 1

a.

$$y = (x^2 - 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} \times 2x$$ 1M

$$= \frac{2x}{2\sqrt{x^2 - 1}}$$

$$= \frac{x}{\sqrt{x^2 - 1}}$$ 1A

b.

$$f'(x) = \frac{vu' - v'u}{v^2}$$ 1M

$$= \frac{1 + \sin x - x\cos x}{(1 + \sin x)^2}$$ 1A

$$f'(\pi) = \frac{1 + \sin \pi - \pi \cos \pi}{(1 + \sin \pi)^2}$$

$$= 1 + \pi$$ 1A

Question 2

a.

$$f(x) = \int \left(x^{\frac{1}{2}} - x^2\right) dx$$ 1M

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} + c$$

$$= \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} + c$$

Since $f(3) = 2$,

$$2 = \frac{2 \times 3^{\frac{3}{2}}}{3} - \frac{3^3}{3} + c$$ 1M

$$2 = 2\sqrt{3} - 9 + c$$

$$c = 11 - 2\sqrt{3}$$

Therefore

$$f(x) = \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} + 11 - 2\sqrt{3}$$ 1A

b.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2\cos(2x) - \sin(3x))\, dx = \left[\sin(2x) - \frac{1}{3}\cos(3x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$ 1M

$$= \sin\left(\frac{2\pi}{3}\right) - \frac{1}{3}\cos(\pi) - \sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\cos\left(\frac{3\pi}{4}\right)$$ 1A

$$= \frac{\sqrt{3}}{2} + \frac{1}{3} - 1 - \frac{\sqrt{2}}{6}$$

$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{6} - \frac{2}{3}$$ 1A

Question 3

a.

$$\frac{dy}{dx} = -\frac{\sin(x)}{\cos(x)}$$ 1M

$$= -\tan(x)$$ 1A

b.

$$\int_0^{\frac{\pi}{3}} 5\tan(x)\, dx = 5\int_0^{\frac{\pi}{3}} \left(-\frac{dy}{dx}\right) dx$$ 1M

$$= -5\int_0^{\frac{\pi}{3}} \frac{dy}{dx}\, dx$$

$$= -5[\log_e(\cos(x))]_0^{\frac{\pi}{3}}$$

$$= -5\left(\log_e\left(\cos\left(\frac{\pi}{3}\right)\right) - \log_e(\cos(0))\right)$$ 1A

$$= -5\left(\log_e\left(\frac{1}{2}\right) - \log_e(1)\right)$$

$$= 5\log_e(2)$$ 1A

Question 4

a.

$$\Pr(-a \leq Z \leq a) = 1 - 2b$$ 1A

b.

$$\Pr(Z \geq -a | Z \leq a) = \frac{\Pr(-a \leq Z \leq a)}{\Pr(Z \leq a)}$$ 1M

$$= \frac{1-2b}{1-b}$$ 1A

Question 5

a. Let X be the total number of heads obtained when the coin is tossed $n = 5$ times. Then $X \sim \text{Bi}\left(5, \frac{1}{2}\right)$. Therefore,

$$\Pr(X \geq 1) = 1 - \Pr(X = 0)$$ 1M

$$= 1 - \left(\frac{1}{2}\right)^5$$

$$= \frac{31}{32}$$ 1A

b.

$$\Pr(X \geq 1 | X \leq 4) = \frac{\Pr(1 \leq X \leq 4)}{\Pr(X \leq 4)}$$ 1M

$$= \frac{1 - \Pr(X = 0) - \Pr(X = 5)}{1 - \Pr(X = 5)}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^5 - \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{30}{31}$$ 1A

Question 6

a.

$$\text{E}(X) = 1$$

$$\frac{1}{2} + \frac{a}{4} + \frac{b}{4} = 1$$

$$\frac{a}{4} + \frac{b}{4} = \frac{1}{2}$$

$$a + b = 2$$ 1A

b.

$$\text{E}(X^2) - \text{E}(X)^2 = 1$$

$$\frac{1}{2} + \frac{a^2}{4} + \frac{b^2}{4} - 1^2 = 1$$ 1M

$$\frac{a^2}{4} + \frac{b^2}{4} = \frac{3}{2}$$

$$a^2 + b^2 = 6$$ 1A

c. Substitute $b = 2 - a$ into $a^2 + b^2 = 6$ to get

$$a^2 + (2 - a)^2 = 6$$ 1M

$$a^2 + 4 - 4a + a^2 = 6$$

$$2a^2 - 4a - 2 = 0$$

$$a^2 - 2a - 1 = 0$$

$$a^2 - 2a + 1 - 2 = 0$$

$$(a - 1)^2 = 2$$

$$a = 1 \pm \sqrt{2}$$

If $a = 1 + \sqrt{2}$ then $b = 1 - \sqrt{2}$, which is impossible since $b > a$. Therefore,

$$a = 1 - \sqrt{2} \text{ and } b = 1 + \sqrt{2}.$$ 1A

Question 7

a. Let $u = 10t$ and $v = e^{-t}$. Therefore,

$$\begin{aligned} C'(t) &= u'v + v'u \\ &= 10e^{-t} - 10te^{-t} \\ &= 10e^{-t}(1-t) \end{aligned}$$

1M

1A

b. The concentration is a maximum when $C'(t) = 0$. Therefore, 1M

$$\begin{aligned} C'(t) &= 0 \\ 10e^{-t}(1-t) &= 0 \\ t &= 1 \end{aligned}$$

1A

since $10e^{-t} \neq 0$.

Question 8

$$\begin{aligned} \Pr(X = 5 \mid X \geq 4) &= \frac{1}{3} \\ \frac{\Pr(X = 5 \text{ and } X \geq 4)}{\Pr(X \geq 4)} &= \frac{1}{3} \\ \frac{\Pr(X = 5)}{\Pr(X = 4) + \Pr(X = 5)} &= \frac{1}{3} \\ \frac{p^5}{{}^5C_4 p^4(1-p) + p^5} &= \frac{1}{3} \\ \frac{p^5}{5p^4(1-p) + p^5} &= \frac{1}{3} \\ \frac{p^5}{p^4(5(1-p) + p)} &= \frac{1}{3} \\ \frac{p}{5-4p} &= \frac{1}{3} \\ 3p &= 5 - 4p \\ p &= \frac{5}{7} \end{aligned}$$

1M

1A

1A

Question 9

a. If $a = 0$, then $f(x) = -4x + 1$ will be one-to-one. Therefore, it has an inverse.

If $a \neq 0$, then the function is a quadratic with turning point

$$x_{TP} = -\frac{b}{2a} = \frac{4}{2a} = \frac{2}{a}.$$

1A

The turning point cannot lie in the interval $(-1,1)$.

If $\frac{2}{a} \leq -1$ then $a \in [-2,0)$.
If $\frac{2}{a} \geq 1$ then $a \in (0,2]$.

Combining the answers, the function will have an inverse for $a \in [-2,2]$. 1A

b. The function f will be one to one provided $f'(x) \geq 0$ for all x. We have, 1M

$$f'(x) = 6x^2 - 2kx + 2$$

We want for $f'(x) = 0$ to have at most one solution. Therefore,

$$\begin{aligned} \Delta &\leq 0 \\ b^2 - 4ac &\leq 0 \\ (-2k)^2 - 48 &\leq 0 \\ k^2 - 12 &\leq 0 \end{aligned}$$

Therefore,

$$-\sqrt{12} \leq k \leq \sqrt{12}$$

1A

EXAM 2, SECTION 1

Question 1 C

Since

$$x_{TP} = -\frac{b}{2a} = -\frac{2k}{2} = -k$$

and

$$y_{TP} = (-k)^2 + 2k(-k) + 3k^2 = 2k^2.$$

Therefore, $2k^2 = 18 \Rightarrow k = \pm 3$.

Question 2 C

Solve

$$\begin{aligned} x &= e^{y-1} + 1 \\ x - 1 &= e^{y-1} \\ y &= 1 + \log_e(x-1) \\ f^{-1}(x) &= 1 + \log_e(x-1) \end{aligned}$$

Also, $\text{dom}(f^{-1}) = \text{ran}(f) = [2, e^2 + 1)$.

Question 3 B

$$x' = -x + 4 \Rightarrow x = 4 - x'$$
$$y' = 3y - 3 \Rightarrow y = \frac{y' + 3}{3}$$

Therefore $y = 4\cos(2x) + 2$ becomes

$$\begin{aligned} \frac{y'+3}{3} &= 4\cos\big(2(4 - x')\big) + 2 \\ y' &= 12\cos\big(2(4 - x')\big) + 3 \end{aligned}$$

Therefore the range is $[3 - 12, 3 + 12] = [-9, 15]$.

Question 4 A

If

$$y = f(2x + 6) = f(2(x + 3))$$

is translated by one unit to the right then the rule becomes

$$y = f\left(2\left((x - 1) + 3\right)\right) = f(2x + 6).$$

Question 5 E

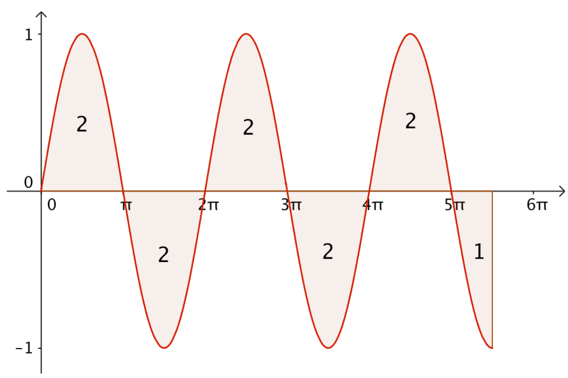
The area above the axis cancels the area beneath the axis. Therefore,

$$\begin{aligned} &\int_0^a 3x^2 - 3\,dx = 0 \\ &[x^3 - 3x]_0^a = 0 \\ &a^3 - 3a = 0 \\ &a(a^2 - 3) = 0 \\ &a = 0, \pm\sqrt{3} \end{aligned}$$

Since $a > 0$, we must have $a = \sqrt{3}$.

Question 6 A

Since $\int_0^\pi \sin x\,dx = 2$, we must have $k = 5.5\pi = \frac{11\pi}{2}$, as shown below.



Question 7 A

Since $A = 1$, we have $\frac{a \times 2}{2} = 1$. Therefore $a = 1$. By symmetry, $\mathrm{E}(X) = \frac{a}{2} = \frac{1}{2}$.

Question 8 B

$$\begin{aligned}\Pr(\text{at least one of each}) &= 1 - \Pr(RRR) - \Pr(BBB)\\ &= 11 - \frac{3}{8}\frac{2}{7}\frac{1}{6} - \frac{5}{8}\frac{4}{7}\frac{3}{6}\\ &= \frac{45}{56}\end{aligned}$$

$$= \frac{45}{56}$$

Question 9 C

Let n be the number of boys in the group. Then

$$\begin{aligned}\Pr(BB) &= \frac{7}{15}\\ \frac{n}{n+3}\frac{n-1}{n+2} &= \frac{7}{15}\end{aligned}$$

Solve for $n > 0$ to get $n = 7$.

Question 10 B

Since

$$\begin{aligned}p + p + 3p + \frac{1}{6} &= 1\\ 5p &= \frac{5}{6}\\ p &= \frac{1}{6}\end{aligned}$$

We have

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = \frac{1}{6} + 1 + \frac{1}{2} = \frac{5}{3}.$$

Question 11 A

Note that f will have an inverse if and only if $f'(x) \geq 0$ for all x. Since $f'(x) = 3x^2 + 2kx + 3$, we require that this quadratic have at most one x-intercept. That is,

$$\Delta = b^2 - 4ac = 4k^2 - 36 \leq 0$$

Therefore $-3 \leq k \leq 3$.

Question 12 A

Since

$$\frac{dy}{dx} = e^{2x}(2x + 2a + 1),$$

and $\frac{dy}{dx} = 0$ when $x = -1$, we have that

$$\begin{aligned}e^{-2}(-2 + 2a + 1) &= 0\\ a &= \frac{1}{2}\end{aligned}$$

Question 13 E

Using similar triangles,

$$\frac{y}{3-x}=\frac{4}{3}\Longrightarrow y=\frac{12-4x}{3}$$

Therefore

$$A=xy=\frac{4}{3}x(3-x)$$

has a maximum at $x=\frac{3}{2}$. Therefore

$$A=\frac{4}{3}\frac{3}{2}\left(3-\frac{3}{2}\right)=3.$$

Question 14 C

$$\Pr(X<4|X>2)=\frac{\Pr(2<X<4)}{\Pr(X>2)}=\frac{1/3}{2/3}=\frac{1}{2}$$

Question 15 C

$$\begin{aligned}2\int_0^{\pi} f(x)\,dx+a\int_0^{\pi}\cos\left(\frac{x}{2}\right)dx&=10\\ 4+a\left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi}&=10\\ 2a\sin\left(\frac{\pi}{2}\right)&=6\\ a&=3\end{aligned}$$

Question 16 A

We have $\frac{dy}{dx}=\frac{2a}{2x-1}$. At $x=1$,

$$m=\frac{dy}{dx}=2a\quad\text{and}\quad y=a\log_e(2-1)+1=1.$$

Therefore the equation of the tangent is

$$y-1=2a(x-1).$$

Let $x=y=0$ so that $-1=-2a$. Therefore $a=\frac{1}{2}$.

Question 17 B

Since $X\sim B\left(15,\frac{3}{4}\right)$ and $\hat{p}=\frac{X}{15}$ we have

$$\begin{aligned}\Pr\left(\hat{p}\geq\frac{2}{3}\right)&=\Pr\left(\frac{X}{15}\geq\frac{2}{3}\right)\\ &=\Pr(X\geq 10)\\ &=0.6865.\end{aligned}$$

Question 18 E

Note that $f'(x) = \frac{a}{2\sqrt{x}} - \frac{b}{2\sqrt{x^3}}$

$$f(1) = 1 \Longrightarrow 1 = a + b$$
$$f'(1) = 0 \Longrightarrow 0 = \frac{a}{2} - \frac{b}{2}$$

Solve these equations to give $(a, b) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Question 19 D

Note that $\text{ran}(g) = [c, \infty)$.
Since $f(x) = x^2 - 2x - 8 = (x-4)(x+2)$, we require that $c > 4$.

Question 20 C

The area of the rectangle is

$$\begin{aligned} A &= uv \\ &= ue^{-u} \\ A' &= -ue^{-u} + e^{-u} \\ &= e^{-u}(1-u) \end{aligned}$$

If $A' = 0$ then $u = 1$ and $A = 1 \times e^{-1} = e^{-1}$.

EXAM 2, SECTION 2

Question 1

a.

$$\int_0^1 \frac{a}{x+1}\,dx = 1$$ 1M

$$a[\log_e(x+1)]_0^1 = 1$$
$$a(\log_e(2) - \log_e(1)) = 1$$ 1A
$$a = \frac{1}{\log_e(2)}$$

b.

$$\text{E}(X) = \int_0^1 \frac{ax}{x+1}\,dx = \frac{1}{\log_e(2)} \int_0^1 \frac{x}{x+1}\,dx = 0.44 \text{ hours}$$ 1A

c.

$$\frac{1}{\log_e(2)}\int_0^m \frac{1}{x+1}dx = \frac{1}{2}$$ 1M

$$m = 0.41 \text{ hours}$$ 1A

d.

$$\frac{1}{\log_e(2)}\int_k^1 \frac{1}{x+1}dx = 0.2$$ 1M

$$k = 0.7411 \text{ hrs}$$
$$= 44.47 \text{ min}$$ 1A

e.

$$\int_0^{0.5} \frac{a}{x+1}dx = 0.58$$ 1M 1A

f.

$$\Pr(X > 0.25|X < 0.5) = \frac{\Pr(0.25 < X < 0.5)}{\Pr(X < 0.5)}$$ 1M

$$= 0.45$$ 1A

g. Let N be the number of times that the wait time is less than 30 minutes. 1M
Then $N \sim B(7,0.58)$ and

$$\Pr(N = 4) = 0.29.$$ 1A

h. $\hat{p} = \frac{764}{2000} = 0.382$ 1A

i. First find k such that

$$\Pr(-k < Z < k) = 0.99$$ 1M
$$\Pr(Z < k) = 0.995$$
$$k = 2.5758$$

Then

$$\hat{\sigma} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.382(1-0.382)}{2000}} = 0.01086$$

Then

$$99\%\ \text{CI} = (\hat{p} - k\hat{\sigma}, \hat{p} + k\hat{\sigma}) = (0.35, 0.41)$$ 1A

Question 2

a. Since

$$f'(x) = \frac{27}{a^4}x(2a - 3x)$$ 1M

if $f'(x) = 0$ then $x = \frac{2a}{3}$ and $f\left(\frac{2a}{3}\right) = \frac{4}{a}$. Therefore, D has coordinates

$$\left(\frac{2a}{3}, \frac{4}{a}\right)$$ 1A

b. Area $= a \times \frac{4}{a} = 4$

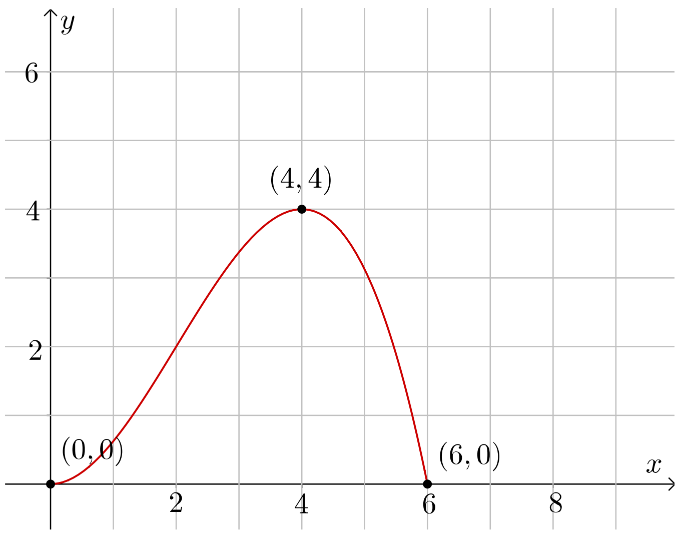
c. Since

$$A = \int_0^a \frac{27}{a^4}x^2(a - x)\,dx = \frac{9}{4}$$ 1A

Therefore,

$$\text{Fraction shaded} = \frac{\frac{9}{4}}{4} = \frac{9}{16}.$$ 1A

d.



shape 1A turning point 1A endpoints 1A

e. The gradient m of the curve is given by

$$f'(x) = \frac{1}{8}(12x - 3x^2)$$ 1M

$$= -\frac{3}{8}(x^2 - 4x)$$
$$= -\frac{3}{8}((x-2)^2 - 4)$$
$$= -\frac{3}{8}(x-2)^2 + \frac{3}{2}$$

The gradient has a maximum at $x = 2$. When $x = 2$, $f(2) = \frac{1}{8}\times 4 \times 4 = 2$. Therefore

$$m = 2 \quad \text{and} \quad n = 2$$ 1A 1A

f. Since

$$h(x) = g(x+2) - 2$$ 1M
$$= \frac{1}{8}(x+2)^2\big(6 - (x+2)\big) - 2$$
$$= \frac{3x}{2} - \frac{x^3}{8}$$ 1A

Therefore,

$$h(-x) = \frac{3(-x)}{2} - \frac{x^3}{8}$$
$$= -\frac{3x}{2} + \frac{x^3}{8}$$ 1A
$$= -h(x)$$

Question 3

a. Since $A \sim \mathrm{N}(502,2)$, 1M

$$\Pr(A < c) = 0.99$$
$$c = 506.65$$ 1A

b. $\Pr(A < 497) = 0.0062$. 1M 1A

c. Since $B \sim \mathrm{N}(\mu, \sigma)$,

$$\Pr(B > 507) = 0.15$$ 1M
$$\Pr(B \leq 507) = 0.85$$
$$\frac{507 - \mu}{\sigma} = 0.85$$

$$\Pr(B < 498) = 0.02$$
$$\frac{498 - \mu}{\sigma} = 0.02$$

Solving these two equations gives 1M

$$\mu = 503.98 \text{ and } \sigma = 2.91$$ 1A

d. $\Pr(B < 497) = 0.0083$ 1M 1A

e.

$$p(0.0062) + (1-p)(0.0083) = 0.007$$ 1M
$$p = 0.619$$
$$\approx 62\%$$ 1A

Question 4

a. $x^2 = \sqrt{x} \Rightarrow x = 1, y = 1$. The coordinates are (1,1). 1A

b. $\int_0^1 (\sqrt{x} - x^2)\, dx = \frac{1}{3}$. 1M 1A

c. $2\times\frac{1}{3} = \frac{2}{3}$ 1A

d. $\sqrt{\frac{1}{4} - \left(\frac{1}{4}\right)^2} = \frac{1}{2} - \frac{1}{16} = \frac{7}{16}$ 1M 1A

e. The vertical separation is given by

$$d(k) = \sqrt{k} - k^2$$ 1M
$$d'(k) = \frac{1}{2\sqrt{k}} - 2k$$

When $d'(k) = 0$ we have 1M

$$\frac{1}{2\sqrt{k}} - 2k = 0$$
$$k = \frac{1}{2^{\frac{4}{3}}}$$ 1A

f. $G = (k, \sqrt{k})$ 1A
When $\sqrt{x} = k^2 \Rightarrow x = k^4$
Therefore, $H = (k^4, k^2)$. 1A

g. $A(k) = \frac{1}{2}(k - k^4)(\sqrt{k} - k^2)$ 1A
Minimise using CAS 1M
$k = 0.53$ 1A

h. Solve 1M

$$y = x^2$$
$$y + x = c$$

for (x, y) using CAS.

$$x = \frac{1}{2}, y = \frac{1}{4}$$

Therefore

1A

$$K = \left(\frac{1}{2}, \frac{1}{4}\right) \text{ and } J = \left(\frac{1}{4}, \frac{1}{2}\right)$$

The area will then be

$$\int_0^{\frac{1}{4}} \left(\sqrt{x} - x^2\right) dx + \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{3}{4} - x - x^2\right) dx = \frac{13}{96}$$ 1A

i. If $J = (k, \sqrt{k})$ then $K = \left(\sqrt{k}, k\right)$ (Since f and g are the inverses of each other) Since the bounded area is $\frac{1}{3}$, half of the the area is $\frac{1}{6}$. Therefore,

$$\int_0^k \sqrt{x} - x^2\, dx + \frac{1}{2}\left(\sqrt{k} + k\right)\left(\sqrt{k} - k\right) - \int_k^{\sqrt{k}} x^2\, dx = \frac{1}{6}$$ 1M

Solving for k gives $k = 0.31467$. Since $(k, \sqrt{k})$ is on the line $x + y = c$, we have, 1A

$$c = \sqrt{k} + k = 0.88$$ 1A